

On V-Semirings and Semirings all of whose Cyclic Semimodules are Injective

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Dedicated to the memory of Professor L. A. Skornjakov

Abstract

In this paper, we introduce and study V- and CI-semirings—semirings all of whose simple and cyclic, respectively, semimodules are injective. We describe V-semirings for some classes of semirings and establish some fundamental properties of V-semirings. We show that all Jacobson-semisimple V-semirings are V-rings. We also completely describe the bounded distributive lattices, Gelfand, subtractive, semisimple, and anti-bounded, semirings that are CI-semirings. Applying these results, we give complete characterizations of congruence-simple subtractive and congruence-simple anti-bounded CI-semirings which solve two earlier open problems for these classes of CI-semirings.

2010 Mathematics Subject Classifications: Primary 16Y60, 16D99, 06A12; Secondary 18A40, 18G05, 20M18

Key words: simple semimodules; injective semimodules; semisimple semirings; V-semirings; CI-semirings; congruence-simple semirings; Morita equivalence of semirings.

⁰The fourth author is partially supported by Vietnam National Foundation for Science and Technology Development (NAFOSTED)

1 Introduction

In the modern homological theory of modules over rings, the results characterizing rings by properties of modules and/or suitable categories of modules over them are of great importance and sustained interest (for a good number of such results one may consult, for example, [2], [33], and [41]). Inspired by this, during the last three decades quite a few results related to this genre have been obtained in different nonadditive settings. Just to mention some of these settings, a very valuable collection of numerous interesting results on characterizations of monoids by properties and/or by categories of acts over them, *i.e.*, results in the so called *homological classification of monoids*, can be found in [29]; and, for results in the *homological classification of distributive lattices*, another non-additive setting, one may consult the survey [8].

Moreover, quite recently there was obtained a number of interesting and important homological results in a more general, and gaining increasing interest, non-additive setting — results in the *homological classification/characterization of semirings* (see, for instance, [21], [22], [16], [17], [18], [24], [25], [19], and the papers cited in them). One may clearly notice a growing interest in developing algebraic and homological theories of semirings and semimodules motivated by their numerous connections with, and applications in, different branches of mathematics, computer science, quantum physics, and many other areas of science (see, for example, [9]). As algebraic objects, semirings are certainly the most natural generalization of such, at first glance different, algebraic systems as rings and bounded distributive lattices. Thus, investigating semirings and their representations, one should undoubtedly use methods and techniques of the ring, lattice and semigroup theories, as well as diverse techniques and methods of categorical and universal algebra. The wide variety of the algebraic techniques involved in studying semirings, and their representations/semimodules, perhaps explains why the research on categorical and homological aspects of theory of semirings and semimodules is still behind that for rings and monoids. In light of this, presenting some new, important and interesting in our view, nontrivial at all, homological considerations, results and techniques to the problems of the homological characterization/classification of semirings, as well as motivating an interest to this direction of research, is a main goal of our paper.

In this paper, we introduce and study semirings with two classes of injective semimodules over them: *V-semirings* [19] — semirings all of whose simple semimodules are injective; and *CI-semirings* — semirings all of whose cyclic semimodules are injective. The investigation and classification of such semirings serves as a fundamental basis to obtain further structural insight of congruence-simple semirings. The paper is organized as follows.

For the reader's convenience, all subsequently necessary and important notions

and facts on semirings and semimodules that cannot be found in [10] and/or [13] are collected in Section 2.

In Section 3, together with constructing some new examples of noncommutative V-semirings, we also characterize V-semirings within important classes of semirings and establish some fundamental properties of V-semirings. Among other results of this section, we single out the following central ones: we give a complete description of semisimple V-semirings (Theorem 3.12); it is shown that the Jacobson-semisimple V-semirings are just the V-rings (Theorem 3.14); it is established that for a semiring to be a V-semiring is a Morita invariant property (Theorem 3.9).

In Section 4, among the main results of the paper are the following ones: we describe the bounded distributive lattices, the Gelfand semirings [10, p. 56], the left subtractive semirings, and the semisimple semirings, that are CI-semirings (Theorem 4.3, Theorem 4.6, Theorem 4.7, and Theorem 4.10, respectively); we give a complete characterization of anti-bounded CI-semirings (Theorem 4.20), essentially generalizing B. Osofsky's celebrated characterization of semisimple rings [38] (see also [34, Theorem 1.2.9] and [33, Corollary 6.47]); and applying Theorem 4.7 and Theorem 4.20, we give a complete description of congruence-simple subtractive CI-semirings (Corollary 4.8) and congruence-simple anti-bounded CI-semirings (Corollary 4.21), respectively.

Finally, all notions and facts of categorical algebra, used here without any comments, can be found in [35]; for notions and facts from semiring theory and universal algebra we refer to [10] and [11], respectively.

2 Preliminaries

2.1 Recall [10] that a *semiring* is a datum $(S, +, \cdot, 0, 1)$ such that the following conditions are satisfied:

- (1) $(S, +, 0)$ is a commutative monoid with identity element 0;
- (2) $(S, \cdot, 1)$ is a monoid with identity element 1;
- (3) Multiplication is distributive over addition from both sides;
- (4) $0s = 0 = s0$ for all $s \in S$.

A semiring that is not a ring we call a *proper semiring*.

As usual, a *left S-semimodule* over the semiring S is a commutative monoid $(M, +, 0_M)$ together with a scalar multiplication $(s, m) \mapsto sm$ from $S \times M$ to M which satisfies the following identities for all $s, s' \in S$ and $m, m' \in M$:

- (1) $(ss')m = s(s'm)$;
- (2) $s(m + m') = sm + sm'$;
- (3) $(s + s')m = sm + s'm$;
- (4) $1m = m$;

$$(5) \ s0_M = 0_M = 0m.$$

Right semimodules over S and homomorphisms between semimodules are defined in the standard manner. And, from now on, let \mathcal{M} be the variety of commutative monoids, and \mathcal{M}_S and ${}_S\mathcal{M}$ denote the categories of right and left S -semimodules, respectively, over a semiring S .

2.2 An element $\infty \in M$ of an S -semimodule M is *infinite* if $\infty + m = \infty$ for every $m \in M$; and $K \leq_S M$ means that K is an S -subsemimodule of M . Also, we will use the following subsets of the elements of an S -semimodule M :

$$\begin{aligned} I^+(M) &:= \{m \in M \mid m + m = m\}; \\ Z(M) &:= \{z \in M \mid z + m = m \text{ for some } m \in M\}; \\ V(M) &:= \{m \in M \mid m + m' = 0 \text{ for some } m' \in M\}; \\ A(M) &:= \{m \in M \mid m + m' + m = m \text{ for some } m' \in M\}. \end{aligned}$$

For a semimodule $M \in |{}_S\mathcal{M}|$, it is obvious that $I^+(M) \cap V(M) = \{0\}$, and $I^+(M) \leq_S Z(M) \leq_S M$. Moreover, if S is an additively regular semiring, then it is easy to see that $I^+(M)$ is a left $I^+(S)$ -semimodule, as well as the subsemimodule $V(M)$ is the largest S -module in M .

A left S -semimodule M is *zeroic* (*zerosumfree*, *additively idempotent*, *additively regular*) if $Z(M) = M$ ($V(M) = 0$, $I^+(M) = M$, $A(M) = M$). In particular, a semiring S is *zeroic* (*zerosumfree*, *additively idempotent*, *additively regular*) if ${}_SS \in |{}_S\mathcal{M}|$ is a zeroic (zerosumfree, additively idempotent, additively regular) semimodule. For example, the *Boolean semiring* $\mathbf{B} = \{0, 1\}$ is a commutative, zeroic, zerosumfree, additively idempotent semiring in which $\infty = 1$.

2.3 A subsemimodule $K \leq_S M$ of a semimodule M is (*strong*) *subtractive* if $(m + m' \in K \Rightarrow m, m' \in K)$ $m, m + m' \in K \Rightarrow m' \in K$ for all $m, m' \in M$. A left S -semimodule M is *subtractive* if it has only subtractive subsemimodules. A semiring S is *left subtractive* if S is a subtractive left semimodule over itself.

As usual (see, for example, [10, Chapter 17]), if S is a semiring, then in the category ${}_S\mathcal{M}$, a *free* (left) semimodule $\sum_{i \in I} S_i$, $S_i \cong {}_SS$, $i \in I$, with a basis set I is a direct sum (a coproduct) of $|I|$ copies of ${}_SS$; a semimodule $P \in |{}_S\mathcal{M}|$ is *projective* if it is a retract of a free semimodule; a semimodule $F \in |{}_S\mathcal{M}|$ is *flat* iff the functor $- \otimes_S F : \mathcal{M}_S \rightarrow \mathcal{M}$ preserves finite limits iff F is a filtered (directed) colimit of finitely generated free (projective) semimodules [21, Theorem 2.10]; a semimodule $M \in |{}_S\mathcal{M}|$ is *finitely generated* (*cyclic*) iff M is a homomorphic image of a free left S -semimodule with a finite basis (a homomorphic image of ${}_SS$); a semimodule $M \in |{}_S\mathcal{M}|$ is *injective* if for any monomorphism $\mu : A \rightarrowtail B$ of left S -semimodules A and B and every homomorphism $f \in {}_S\mathcal{M}(A, M)$, there exists a homomorphism $\tilde{f} \in {}_S\mathcal{M}(B, M)$ such that $\tilde{f}\mu = f$.

2.4 *Congruences* on an S -semimodule M are defined in the standard manner, and $\text{Cong}(M)$ denotes the set of all congruences on M . This set is non-empty

since it always contains at least two congruences—the *diagonal congruence* $\Delta_M := \{(m, m) \mid m \in M\}$ and the *universal congruence* $M^2 := \{(m, n) \mid m, n \in M\}$. Any subsemimodule $L \leq_S M$ of an S -semimodule M induces a congruence \equiv_L on M , known as the *Bourne congruence*, by setting $m \equiv_L m'$ iff $m + l = m' + l'$ for some $l, l' \in L$; and M/L denotes the factor S -semimodule M/\equiv_L having the canonical S -surjection $\pi_L : M \rightarrow M/L$.

Furthermore, a nonzero S -semimodule M is *simple* (*atom*, *s-simple*) if $\text{Cong}(M) = \{\Delta_R, M^2\}$ (M has no nonzero proper S -subsemimodules, M has no nonzero proper subtractive S -subsemimodules). The following observations will prove to be useful.

Proposition 2.5 *For a nonzero S -semimodule M the following statements are true:*

- (1) *M is simple iff every nonzero semimodule homomorphism $f : M \rightarrow N$ is injective;*
- (2) *If M is simple, then M is s-simple.*

Proof (1) \implies . Let $f : M \rightarrow N$ be a nonzero morphism and \equiv_f the congruence on M defined by $m \equiv_f m'$ iff $f(m) = f(m')$. Since M is a simple semimodule and f is a nonzero homomorphism, it is easy to see that $\equiv_f = \Delta_M$ and, hence, f is injective.

\impliedby . It is obvious.

- (2) See [10, Proposition 15.13]. \square

3 On V-semirings

Generalizing the well known for rings notions and following [19], we call a semiring S a *left (right) V-semiring* if every simple left (right) S -semimodule is injective; and an S -semimodule M is called an *essential extension* of an S -subsemimodule $L \leq_S M$, $i : L \rightarrow M$, if for every semimodule homomorphism $\gamma : M \rightarrow N$, the homomorphisms γi and γ are simultaneously injective. The following characterization of V-semirings will prove to be useful.

Theorem 3.1 ([19, Theorem 2.10]) *The following statements for a semiring S are equivalent:*

- (1) *S is a left (right) V-semiring;*
- (2) *Every essential extension of each simple left (right) S -semimodule M coincides with M ;*
- (3) *$S \cong R \oplus Z$, where R is a left (right) V-ring and Z is a zeroic left (right) V-semiring;*
- (4) *Every quotient semiring of S is a left (right) V-semiring.*

In particular, one may easily see that item (2) of this characterization implies

Corollary 3.2 *Finite direct products of left (right) V-semirings are left (right) V-semirings.*

In this section, in addition to constructing some new examples of noncommutative V-semirings, we also characterize V-semirings within important classes of semirings, and establish some fundamental properties of V-semirings.

From Theorem 3.1 it follows that for a zerosumfree V-semiring to be zeroic is, in general, only a necessary condition. However, it is also a sufficient condition if the semiring has only two trivial strong left (right) ideals.

Proposition 3.3 *A zerosumfree semiring S possessing only two strong left (right) ideals is a left (right) V-semiring iff it is a zeroic semiring.*

Proof \implies . It follows immediately from Theorem 3.1 (3).

\impliedby . Let M be an arbitrary simple left S -semimodule. By [19, Proposition 1.2], $(M, +, 0)$ is either a group or an idempotent monoid. Consider these two cases.

Let $(M, +, 0)$ be a group. Since S is zeroic, there exists $z \in S$ such that $1 + z = z$, and, hence, $zm = (1 + z)m = 1m + zm$ and $m = 1m = 0_M$ for every $m \in M$, what contradicts $M \neq 0$.

Now, let $(M, +, 0)$ be an idempotent monoid. For every $0 \neq m \in M$, the left annihilator $(0 :_S m) := \{ s \in S \mid sm = 0_M \}$ is obviously a strong left ideal of S . By [19, Lemma 1.1], there exists a congruence $\sigma_M \in \text{Cong}(M)$ on M defined by

$$m \sigma_M m' \iff (0 :_S m) = (0 :_S m').$$

Since ${}_S M$ is simple, $\sigma_M = \Delta_M$ and, since $\{0\}$ and S are the only strong left ideals of S , the annihilator $(0 :_S m) = 0$ for every $0 \neq m \in M$ and, therefore, $M = \{0, m\}$. Let $N \in |\mathcal{S}\mathcal{M}|$ be an essential extension of M with the canonical injection $i : M \hookrightarrow N$, and consider the congruence $\diamond_N \in \text{Cong}(N)$ on N defined by

$$n_1 \diamond_N n_2 \iff l_1 n_1 = n_2 + n'_2 \text{ \& } l_2 n_2 = n_1 + n'_1 \text{ for some } l_1, l_2 \in \mathbb{N}, n'_1, n'_2 \in N.$$

By [19, Lemma 2.2], $N^\diamond := N/\diamond_N$ is an additively idempotent semimodule with the canonical surjection $p : N \twoheadrightarrow N^\diamond$. As $m \in I^+(M)$, it is clear that $(m, 0) \notin \diamond_N$ and therefore the map $pi : M \hookrightarrow N \twoheadrightarrow N^\diamond$ is injective. Since N is an essential extension of M , one has that p is an injective surjection, *i.e.*, p is an isomorphism and N is an additively idempotent semimodule. Then, considering the congruence $\sigma_N \in \text{Cong}(N)$ on N and using the same arguments as above for the semimodule M , one sees that $N/\sigma_N = \{\bar{0}, \bar{m}\}$ with the natural surjection $\pi : N \twoheadrightarrow N/\sigma_N$. Noting that the map $\pi i : M \twoheadrightarrow N \twoheadrightarrow N/\sigma_N$ is injective and N is an essential extension of M , one concludes that π is an isomorphism, $M = N$, and, applying Theorem 3.2 (2), ends the proof. \square

It is easy to see that $V(S)$ is a strong left and right ideal in a semiring S . From this observation, Theorem 3.1 (3) and Proposition 3.3, we have

Corollary 3.4 *A semiring with only two strong left (right) ideals is a left (right) V-semiring iff it is either a left (right) V-ring, or a zeroic proper semiring.*

It is obvious that in any proper division semiring there are only two trivial strong left (right) ideals, and therefore from Corollary 3.4 we obtain

Corollary 3.5 *A division semiring is a left (right) V-semiring iff it is either a zeroic proper division semiring or a division ring.*

Now, we introduce a quite interesting class of semirings, naturally extending the class of all rings. For any semiring S , let $P(S) := V(S) \cup \{1 + s \mid s \in S\}$. It is easy to see that $P(S)$ is always a subsemiring of S ; and when $P(S) = S$, we say that the semiring S is *anti-bounded*.

From Proposition 3.3 we immediately have our first observation about anti-bounded semirings.

Corollary 3.6 *A zerosumfree anti-bounded semiring S is a left (right) V-semiring iff it is zeroic.*

Proof Indeed, a zerosumfree anti-bounded semiring S has only two strong left (right) ideals: If an ideal $I \neq 0$ is a strong left (right) ideal of S , $1 + s \in I$ for some $s \in S$ and, hence, $1 \in I$. \square

Clearly, proper division semirings are zerosumfree semirings containing only two strong left (right) ideals. However, the class of zerosumfree semirings possessing only two strong left (right) ideals, as the following example shows, is quite wider than the class of proper division semirings.

Example 3.7 Let n be a nonzero natural number and \mathbf{B}_{n+1} the join-semilattice defined on the chain $0 < 1 < \dots < n$. Equip \mathbf{B}_{n+1} with a structure of a semiring with addition $x + y := x \vee y$ and multiplication

$$xy := \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0 \\ x \vee y, & \text{otherwise} \end{cases}.$$

It is easy to see that \mathbf{B}_{n+1} is a zerosumfree anti-bounded zeroic semiring with infinite element $\infty = n$, that also is, by Corollary 3.6, a V-semiring and is not a division semiring. Of course, \mathbf{B}_2 coincides with Boolean semiring \mathbf{B} .

Recall (see, for example, [22] and [24]) that two semirings T and S are said to be *Morita equivalent* if the semimodule categories ${}_T\mathcal{M}$ and ${}_S\mathcal{M}$ are equivalent categories; *i.e.*, there exist two additive functors $F : {}_T\mathcal{M} \longrightarrow {}_S\mathcal{M}$ and $G : {}_S\mathcal{M} \longrightarrow {}_T\mathcal{M}$, and natural isomorphisms $\eta : GF \longrightarrow Id_{{}_T\mathcal{M}}$ and $\xi : FG \longrightarrow Id_{{}_S\mathcal{M}}$. By [24, Theorem 4.12], two semirings T and S are Morita equivalent iff the

semimodule categories \mathcal{M}_T and \mathcal{M}_S are equivalent categories. Following [24], a left semimodule ${}_T P \in |{}_T \mathcal{M}|$ is said to be a *generator* in the category ${}_T \mathcal{M}$ if the regular semimodule ${}_T T \in |{}_T \mathcal{M}|$ is a retract of a finite direct sum $\oplus_i P$ of the semimodule ${}_T P$; and a left semimodule ${}_T P \in |{}_T \mathcal{M}|$ is said to be a *progenerator* in ${}_T \mathcal{M}$ if it is a finitely generated projective generator. Finally, by [24, Theorem 4.12] two semirings T and S are Morita equivalent iff there exists a progenerator ${}_T P \in |{}_T \mathcal{M}|$ in ${}_T \mathcal{M}$ such that the semirings S and $\text{End}({}_T P)$ are isomorphic.

Our next observation is that the classes of simple and injective semimodules are preserved by Morita equivalences of semirings, more precisely:

Lemma 3.8 *Let $F : {}_T \mathcal{M} \rightleftarrows {}_S \mathcal{M} : G$ be an equivalence of the semimodule categories ${}_T \mathcal{M}$ and ${}_S \mathcal{M}$. Then, a semimodule $M \in |{}_T \mathcal{M}|$ is simple (injective) iff $F(M) \in |{}_S \mathcal{M}|$ is simple (injective).*

Proof By the dual of [24, Lemmas 4.7 and 4.10], one easily has that a semimodule $M \in |{}_T \mathcal{M}|$ is injective if and only if $F(M) \in |{}_S \mathcal{M}|$ is injective.

Now, let $M \in |{}_T \mathcal{M}|$ be a simple semimodule, and $f : F(M) \rightarrow N$ be a nonzero homomorphism in ${}_S \mathcal{M}$. Then, $G(f) : M \cong G(F(M)) \rightarrow G(N)$ is a nonzero homomorphism in ${}_T \mathcal{M}$, and, hence, it is injective. Applying the functor F and the dual of [24, Lemmas 4.7], we have

$$F(G(f)) : F(M) \cong F(G(F(M))) \rightarrow F(G(N)) \cong N$$

is injective, and, hence, f is injective, too; and by Proposition 2.5, $F(M)$ is a simple semimodule, too. \square

Applying Lemma 3.8, we immediately establish that for semirings to be a left (right) V-semiring is a Morita invariant property.

Theorem 3.9 *Let T and S be Morita equivalent semirings. Then T and S are left (right) V-semiring simultaneously.*

Following [3], we call a semiring S *congruence-simple* (*ideal-simple*) if the diagonal and universal congruences are the only congruences on S (if 0 and S are the only ideals of S); finally, a semiring S is said to be *simple* if it is simultaneously congruence- and ideal-simple. In contrast to the varieties of groups and rings, the research on simple semirings has just recently been started, and therefore not too much is so far known on them (for some recent activity and results on this subject, one may consult [3], [37], [4], [42], [20], [25], [28]). Thus, regarding relations between V-semirings and different variations of ‘simplicity’ of semirings, the following observations deserve to be mentioned.

Corollary 3.10 *A simple semiring S with either an infinite element or a projective minimal one-sided ideal is a left (right) V-semiring.*

Proof For a semiring S with an infinite element, the result immediately follows from Example 3.7, Theorem 3.9 and [27, Theorem 5.7].

If a semiring S possesses a projective minimal one-sided ideal, then the result follows from Example 3.7, Theorem 3.9 and [27, Theorems 5.7 and 5.11]. \square

Using the complete description of ideal-simple and congruence-simple Artinian subtractive semirings given in [25, Theorems 3.7 and 4.5], one has

Corollary 3.11 *An ideal-simple left (right) Artinian, left (right) subtractive semiring S is a left (right) V-semiring iff $S \cong M_n(D)$ for some division ring D , or S is a zeroic division semiring. Any congruence-simple left (right) Artinian, left (right) subtractive semiring is a left (right) V-semiring.*

Proof The first statement follows right away from [25, Theorem 3.7], Corollary 3.5, Theorem 3.9, [22, Theorem 5.14], and the classical fact — over division rings all modules are injective.

The second statement also follows right away from [25, Theorem 4.5], Example 3.7, [22, Theorem 5.14], and the same classical fact. \square

As usual, a semiring S is said to be *left (right) semisimple* if the regular semi-module is a direct sum of left (right) atom ideals. Recall (see, for example, [13, Theorem 7.8], or [23, Theorem 4.5]) that a semiring S is (left, right) *semisimple* iff

$$S \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r), \quad (*)$$

where $M_{n_1}(D_1), \dots, M_{n_r}(D_r)$ are the semirings of $n_1 \times n_1, \dots, n_r \times n_r$ matrices for suitable division semirings D_1, \dots, D_r and positive integers n_1, \dots, n_r . In the sequel, we will refer to such an isomorphism $(*)$ as a *direct product representation* of a semisimple semiring S .

Our next result provides us with a description of semisimple left (right) V-semirings:

Theorem 3.12 *The following conditions for a semisimple semiring S are equivalent:*

- (1) *S is a left (right) V-semiring;*
- (2) *$S \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$, where D_1, \dots, D_r are either division rings or zeroic division semirings.*

Proof (1) \implies (2). Let S be a left V-semiring and $S \simeq M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$ a direct product representation of S . By Theorem 3.1, $M_{n_i}(D_i)$, $i = 1, \dots, r$, are left V-semirings, too. Whence, by [22, Theorem 5.14] and Theorem 3.9, D_i , $i = 1, \dots, r$, are also left (right) V-semiring, and applying Corollary 3.5 one gets the statement.

(2) \implies (1). This implication follows straight away from Corollaries 3.2, 3.5, [22, Theorem 5.14] and Theorem 3.9. \square

Corollary 3.13 *Every additively regular, in particular every finite, semisimple semiring is a left (right) V-semiring.*

Proof First, let $S \simeq M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$ be a direct product representation of an additively regular semisimple semiring S . By Corollary 3.2, it is easy to see that, without loss of generality, we can consider only the case when all D_i , $i = 1, \dots, r$, are proper additively regular and, therefore, even additively idempotent (see also [25, p. 4349]), division semirings. From the latter, Corollary 3.5, Theorem 3.9, and [22, Theorem 5.14] one gets that all semirings $M_{n_i}(D_i)$, $i = 1, \dots, r$, are left (right) V-semirings, too, and our assertion follows from Corollary 3.2.

If S is a finite semisimple semiring, then all division semirings D_i , $i = 1, \dots, r$, are also finite. Then, applying, for instance, [14, Proposition 1.2.3], one gets that the monogenic additive subsemigroup $\langle 1_i \rangle$ of each monoid $(D_i, +, 0)$ generated by $1_i \in D_i$, $i = 1, \dots, r$, contains a nonzero idempotent, and therefore, each division semiring D_i , $i = 1, \dots, r$, is additively idempotent. So, S is also an additively idempotent semiring, and the statement follows from the previous one above. \square

We conclude this section by considering some relations between V-semirings and the *Jacobson-Bourne radical for semirings* — a semiring analog and/or generalization of the classical Jacobson radical for rings — introduced by S. Bourne in [5].

Recall [5] that a right ideal I of a semiring S is said to be *right semiregular* if, for every pair of elements i_1, i_2 in I , there exist elements j_1 and j_2 in I such that $i_1 + j_1 + i_1 j_1 + i_2 j_2 = i_2 + j_2 + i_1 j_2 + i_2 j_1$. As was shown in [5, Theorems 3 and 4], the sum of all right semiregular ideals of a semiring S , denoted by $J(S)$, is also a right semiregular ideal of S . This ideal is called the *Jacobson radical* of the semiring S , and S is said to be *Jacobson-semisimple* if $J(S) = 0$. As was shown in [33, Theorem 3.75], all left (right) V-ring S are Jacobson-semisimple rings, *i.e.*, $J(S) = 0$. However, this is not true for V-semirings in general: For example, by Corollary 3.5, the Boolean semifield \mathbf{B} is a left (right) V-semiring, but it is very easy to see that $J(\mathbf{B}) = \mathbf{B}$ (see also [26, Example 3.7]). In light of this fact, our next result shows that the class of the Jacobson-semisimple V-semirings contains only rings, namely:

Theorem 3.14 *A left (right) V-semiring S is Jacobson-semisimple iff S is a left (right) V-ring.*

Proof \Rightarrow . Let S be a Jacobson-semisimple left (right) V-semiring. Then, by [26, Corollary 4.6], S is semiisomorphic to a subdirect product of some additively cancellative semirings $\{S_\lambda\}_{\lambda \in \Lambda}$ whose rings of differences S_λ^Δ (see, [10, p. 101]) are isomorphic to dense subrings of the rings of linear transformations $\text{End}({}_{D_\lambda}V_\lambda)$ of vector spaces ${}_{D_\lambda}V_\lambda$ over division rings D_λ for each $\lambda \in \Lambda$. Since all semirings S_λ , $\lambda \in \Lambda$, are additively cancellative, by Theorem 3.1 (4), they are clearly left (right) V-rings, and, hence, S is a left (right) V-ring, too.

\Leftarrow . This follows from [33, Theorem 3.75]. \square

There is another very natural semiring analog of the Jacobson radical for semirings, J_s -radical, based on the class of simple semimodules, considered in [26], and is, in general, different from J -radical ([26, Example 3.7]). It is easy to see that a proper division semiring D is a left (right) V -semirings with $J_s(D) = 0$, and hence, an analog of Theorem 3.14 for the J_s -semisimple semirings is not true. In light of this, we end this section by posing, in our view, an interesting and perspective question.

Problem 1. Describe all J_s -semisimple V -semirings.

4 Semirings all of whose cyclic semimodules are injective

Inspired by the well-known characterization of semisimple rings as the rings all of whose cyclic modules are injective given by Barbara Osofsky ([38], or [33, Corollary 6.47])), we initiate in this section a study of the so called *left (right) CI-semirings* — semirings all of whose cyclic semimodules are injective. As our next observation shows, the class of CI-semirings is significantly wider than that of semisimple rings; and the CI-semirings, in our view, will constitute a very interesting direction of future investigations in the non-additive semiring setting. Here, by characterizing the CI-semirings within some special and important classes of semirings, we are just starting, and hopefully motivating too, the research in this promising direction.

Proposition 4.1 *A semiring S is a left (right) CI-semiring iff $S = R \oplus T$ with R and T a semisimple ring and a zerosumfree left (right) CI-semiring with an infinite element, respectively.*

Proof \Rightarrow . Let S be a left CI-semiring, $\equiv_{V(S)}$ the Bourne congruence on S . It is clear that the factor semiring $\overline{S} := S / \equiv_{V(S)}$ is zerosumfree, and the natural surjection $\pi : S \rightarrow \overline{S}$ induces the *restriction functor* $\pi^\# : \overline{S}\mathcal{M} \rightarrow {}_S\mathcal{M}$ (see [22, p. 202, Proposition 4.1]). The semiring \overline{S} is a left CI-semiring too: Indeed, if $M \in |\overline{S}\mathcal{M}|$ is a cyclic \overline{S} -semimodule, then $\pi^\#(M) \in |{}_S\mathcal{M}|$ is a cyclic S -semimodule as well and, hence, injective; for [24, Lemma 5.2], the latter implies that M is an injective \overline{S} -semimodule as well. In particular, for the zerosumfree semiring \overline{S} , the regular semimodule $\overline{S} \in |\overline{S}\mathcal{M}|$ is injective, and therefore, by [18, Proposition 1.7], the semiring \overline{S} contains an infinite element and, hence, it is a zeroic zerosumfree semiring. So, by [19, Proposition 2.9], $S = R \oplus T$, where R is a ring and T is a semiring isomorphic to \overline{S} . One readily sees that T is a left CI-semiring with an infinite element and the ring R is a left CI-ring. Therefore, by [38] (see also [34, Theorem 1.2.9] or [33, Corollary 6.47])), R is a semisimple ring.

\Leftarrow . It is easy to see that a finite direct product of left CI-semirings is also a left CI-semiring. Using this observation and Osofsky's result [38], one ends the proof. \square

From this proposition one sees right away that the problem of describing CI-semirings is actually reduced to the quite nontrivial problem of describing all zerosumfree left (right) CI-semirings with infinite elements. One important subclass of the class of zerosumfree left (right) CI-semirings with infinite elements — the class of distributive lattices with 0 and 1 (*i.e.*, bounded distributive lattices) that are CI-semirings — have been already considered in Dr. Kornienko's PhD thesis, supervised by late Prof. L. A. Skornjakov and successfully defended at Moscow State University (MGU), USSR, in 1979 (see also the announcement of the results in [30, p. 118]). However, since a proof of Dr. Kornienko's result, to the best of our knowledge, has never been published in any publicly available publications, we find it reasonable to present our independent proof of that result here, too. First, we need the following easy fact:

Lemma 4.2 (*cf.* [40, Theorem 32]) *Any infinite Boolean algebra B has a countable set of orthogonal idempotents.*

Proof Let $B = (B; +, \cdot, ', 0, 1)$ be an infinite Boolean algebra. Take an element $a \in B$ such that $0 \neq a \neq 1$; obviously, $0 \neq a' \neq 1$ and $B = aB \oplus a'B$. It is clear that aB and $a'B$ are also Boolean algebra and at least one of them, say $B_1 = a'B$, is infinite. Let $e_1 := a$. Repeating the same procedure for the algebra $B_1 = e_1'B$, one gets a nonzero element $e_2 \in B_1$ and an infinite subalgebra $B_2 = e_2'B_1$. Continuing this process, one readily gets a countable sequence $\{e_1, e_2, \dots\}$ of mutually orthogonal idempotents of B . \square

The next result, announced in [30, p. 118], characterizes the CI-semirings within the class of bounded distributive lattices.

Theorem 4.3 *A bounded distributive lattice is a CI-semiring iff it is a finite Boolean algebra.*

Proof \Rightarrow . Let a CI-semiring B be a bounded distributive lattice. By [31, Theorem], the lattice B is a complete Boolean algebra. It is clear that a factor algebra of a Boolean algebra as well as a factor semiring of a CI-semiring are a Boolean algebra and a CI-semiring, respectively, too. Therefore, it is enough to show that any infinite complete Boolean algebra has a non-complete factor algebra. Thus, assume that the lattice B is an infinite complete Boolean algebra with, by Lemma 4.2, a countable sequence $\{e_1, e_2, \dots\}$ of mutually orthogonal nonzero idempotents. We shall show that the factor algebra $\overline{B} := B/A$ of the algebra B with respect to the ideal $A := \sum_{i=1}^{\infty} e_i B$ is not a complete Boolean algebra.

Consider $I_p := \{ p^k \mid k = 1, 2, \dots \}$ and $f_p := \bigvee_{i \in I_p} e_i \in B$ for every prime number p . Certainly, the set $\overline{F} = \{ \overline{f_p} \mid p \text{ is prime} \}$ of the elements of the Boolean algebra \overline{B} has an upper bound (for instance, the element $\overline{1}$ is its upper bound); however, as we will show, there is no supremum of \overline{F} — for every upper bound \overline{f} of \overline{F} , there exists another upper bound \overline{g} of the set \overline{F} such that $\overline{g} < \overline{f}$. Indeed, from the inequality $\overline{f_p} \leq \overline{f}$ for every prime p , it follows that there are some elements $a_{p1}, a_{p2} \in A$ such that $\bigvee_{i \in I_p} e_i + f + a_{p1} = f_p + f + a_{p1} = f + a_{p2}$. Whence, multiplying the latter equation by e_i , $i \in I_p$, and using the join infinite distributive identity for complete Boolean algebra [12, Lemma 2.4.10] and the mutual orthogonality of the idempotents e_i , $i = 1, 2, \dots$, we have $e_i = f e_i + a_{p2} e_i$ and $a_{p2} e_i = 0$ for all but finite $i \in I_p$. From the latter, $e_i = f e_i$ for all but finite $i \in I_p$; therefore, for the index set $J_p := \{ i \in I_p \mid e_i = f e_i \}$ we have $|I_p \setminus J_p| < \infty$ and $\overline{f_p} = \overline{g_p}$ for $g_p := \bigvee_{i \in J_p} e_i$.

Now, let $g := \bigvee_p g_p$. Since $\overline{f_p} = \overline{g_p} \leq \overline{g}$ for each p , \overline{g} is an upper bound of the set \overline{F} and, by construction, $\overline{g} \leq \overline{f}$. If the latter inequality is a strong one, \overline{g} is exactly the element we are looking for. So, suppose that $\overline{g} = \overline{f}$, and for each prime p consider the index sets $K_p := J_p \setminus \{j_p\}$, where j_p is the smallest element of the index set J_p . For $|I_p \setminus K_p| < \infty$, we have $\overline{f_p} = \overline{h_p}$ for $h_p := \bigvee_{i \in K_p} e_i$ and $\overline{f_p} = \overline{h_p} \leq \overline{h}$ for $h := \bigvee_p h_p$; whence, \overline{h} is an upper bound of \overline{F} . Moreover, $g = h + u$, where $u := \bigvee_p e_{j_p}$ and, applying again [12, Lemma 2.4.10], $hu = 0$; therefore, $\overline{g} = \overline{h} + \overline{u}$ and $\overline{h}\overline{u} = \overline{0}$. Whence, $\overline{h} < \overline{g}$ provided $\overline{u} \neq \overline{0}$. However, the latter is almost obvious: Indeed, there are infinite number of prime numbers p , all j_p are different for different prime numbers; whence, $u \notin A$ and $\overline{u} \neq \overline{0}$.

\Leftarrow . This immediately follows from [7, Theorem 4]. \square

Combining [31, Corollary 8], [32, Theorem], and Theorem 4.3, one obtains the following characterization of finite Boolean algebras among bounded distributive lattices:

Corollary 4.4 *The following conditions for a bounded distributive lattice B are equivalent:*

- (1) *All cyclic B -semimodules are projective;*
- (2) *All atom B -semimodules are projective;*
- (3) *All subsemimodules of the regular semimodule B_B are injective;*
- (4) *B is a CI-semiring;*
- (5) *B is a finite Boolean algebra.*

As usual, we denote by $U(S)$ the set of all units of a semiring S and, following [10, p. 56], we say that a semiring S is a *Gelfand semiring* if the element $1 + s \in U(S)$ for every element $s \in S$. Of course, bounded distributive lattices are among Gelfand semirings. But the class of the Gelfand semirings is quite wider as the following example, for instance, shows:

Example 4.5 [10, Example 4.49] Let A be a nonempty set having more than one element and $S = (\mathbb{R}^+)^A$ the set of all functions from A to \mathbb{R}^+ , where \mathbb{R}^+ is the set of all nonnegative real numbers, with the canonical semiring structure defined on it. Obviously, $U(S) = \{f \in S \mid f(a) > 0 \text{ for all } a \in A\}$ and S is a Gelfand semiring that is not a bounded distributive lattice.

In this connection, the following characterization of CI-, Gelfand semirings is certainly of interest.

Theorem 4.6 *A Gelfand semiring S is a left (right) CI-semiring iff it is a finite Boolean algebra.*

Proof \Rightarrow . Let S be a left CI-semiring. By Proposition 4.1, $S = R \oplus T$ with R a semisimple ring and T a left CI-semiring with an infinite element ∞ . Let $1 = r + t$ for some $r \in R$ and $t \in T$. Since S is a Gelfand semiring, $t = 1 - r \in U(S)$ and, by [10, Proposition 4.50], $\infty = t + \infty \in U(S)$. As $\infty + \infty^2 = \infty$, we have $\infty = 1 + \infty = 1$, and therefore $S = T$ is an additively idempotent semiring.

Consequently, there is a natural partial order on the semiring S given by $a \leq b$ iff $a + b = b$ with respect to which $a \vee b = a + b$ for all $a, b \in S$. Since S is a CI-semiring, by [1, Theorem 2.2], S is also a von Neumann regular semiring, and hence, for any $s \in S$ there exists $x \in S$ such that $s = sxs$. The latter implies that $s + s^2 = sxs + s^2 = s(x+1)s = s^2$ and $s \leq s^2$. On the other hand, as $s \leq 1$ one has $s^2 \leq s$ and therefore $s = s^2$ for all $s \in S$. Moreover, S is a commutative semiring: Indeed, since $a \leq 1$ and $b \leq 1$ for all $a, b \in S$, it follows that $ab \leq b$, $ab \leq a$ and $ab = (ab)^2 = (ab)(ab) \leq ba$, and by symmetry, $ba \leq ab$ and, hence, $ab = ba$. We show that $a \wedge b = ab$ for all $a, b \in S$: First, $a + ab = a(1 + b) = a \cdot 1 = a$ and $b + ab = (1 + a)b = 1 \cdot b = b$, and hence, $ab \leq a$, $ab \leq b$, and $ab \leq a \wedge b$. Next, if for some $x \in S$ we have $x \leq a$ and $x \leq b$, then

$$\begin{aligned} ab &= (x + a)(x + b) &= x^2 + xb + ax + ab \\ &= x + xb + xa + ab &= x(1 + b + a) + ab \\ &= x \cdot 1 + ab &= x + ab, \end{aligned}$$

i.e., $x \leq ab$, and therefore, $a \wedge b = ab$. Thus, our semiring S is, in fact, a bounded distributive lattice (S, \vee, \wedge) , and applying Theorem 4.3 one gets the statement.

\Leftarrow . This is true by Theorem 4.3. \square

In light of these results, the following problem, we believe, seems to be quite natural and interesting.

Problem 2. To which extent can the equivalent conditions of Corollary 4.4 be extended to Gelfand semirings?

Also, in the connection with a description of the bounded distributive lattices that are V-semirings obtained in [19, Corollary 3.11], it is natural to post

Problem 3. Describe Gelfand V-semirings.

Next, using Theorem 4.6, we give a complete description of left subtractive left CI-semirings.

Theorem 4.7 *A left subtractive semiring S is a left CI-semiring iff $S = R \oplus T$ with R and T a semisimple ring and a finite Boolean algebra, respectively.*

Proof \implies . By Proposition 4.1, $R = S \oplus T$, where S is a classical semisimple ring, and T is a zerosumfree left CI-semiring with the infinite element ∞ . It is easy to see (e.g., by [23, Lemma 4.7]) that T is a left subtractive semiring, too. Then, the left ideal $T\infty$ is subtractive in T and it follows from $1_T + \infty = \infty$ that $1_T \in T\infty$, that means, $t\infty = 1_T$ for some $t \in T$.

On the other hand, we have $\infty^2 + \infty = \infty$. This implies that

$$\infty = \infty + 1_T = 1_T,$$

therefore, T is a Gelfand semiring. By Theorem 4.6, T is a finite Boolean algebra.

\Leftarrow . The semirings R and T are, obviously, left subtractive semirings. By [23, Lemma 4.7], $S = R \oplus T$ is a left subtractive semiring, too.

Applying Proposition 4.1 and Theorem 4.6, we conclude that S is a left CI-semiring. \square

By [3, Theorem 14.1] and [25, Corollary 4.6], the concepts of congruence-simpleness and ideal-simpleness coincide for finite commutative and finite left (right) subtractive semirings. Also, using Theorem 4.6 and [25, Theorem 3.7], we have the same situation for subtractive CI-semirings and, in fact, solve [25, Problem 3] and [27, Problem 5] in the class of subtractive CI-semirings.

Corollary 4.8. *For a left subtractive semiring S , the following conditions are equivalent:*

- (i) *S is a congruence-simple left CI-semiring;*
- (ii) *S is an ideal-simple left CI-semiring;*
- (iii) *$S \cong M_n(D)$ for some division ring D , or $S \cong \mathbf{B}$.*

Proof (i) \implies (ii). It follows immediately from [25, Proposition 4.4].

(ii) \implies (iii). Assume that S is an ideal-simple left CI-semiring. Applying Theorem 4.7, S is either a semisimple ring, or a finite Boolean algebra. If S is a semisimple ring, then since S is ideal-simple, $S \cong M_n(D)$ for some division ring D and $n \geq 1$. Otherwise, S is a finite Boolean algebra without proper nonzero ideals; therefore, S is just the Boolean semifield \mathbf{B} .

(iii) \implies (i). It follows immediately from Proposition 4.1, Theorem 4.7, and [25, Theorem 4.5]. \square

Now, let us consider semisimple CI-semirings. Using the direct product representation $(*)$ of semisimple semirings, our considerations are naturally reduced to the ones of the matrix CI-semirings over division semirings about which the following observation is crucial.

Proposition 4.9 *The matrix semiring $S = M_n(D)$ over a division semiring D is a left (right) CI-semiring iff D is a division ring, or $D \cong \mathbf{B}$ and $n = 1, 2$.*

Proof We will use the equivalence of the semimodule categories ${}_S\mathcal{M}$ and ${}_D\mathcal{M}$ established in [22, Theorem 5.14]: $F : {}_S\mathcal{M} \rightleftharpoons {}_D\mathcal{M} : G$, $F(A) = E_{11}A$ and $G(B) = B^n$, where E_{11} is the matrix unit in $M_n(D)$.

\implies . Let $S = M_n(D)$ be a left CI-semiring. By [1, Theorem 2.2], S is a von Neumann regular semiring, and, if $n \geq 3$, by [15, Proposition 2], D is a regular ring. Hence, we need to consider only the case with $n = 1, 2$.

Let D be a zerosumfree division semiring. Since ${}_D D \cong F(G({}_D D))$, $G({}_D D) = {}_S D^2$, and noting that ${}_S D^2$ is obviously a cyclic S -semimodule and, therefore, an injective one, by Lemma 3.8, one has that ${}_D D$ is an injective D -semimodule. Whence, by [18, Proposition 1.7], D has an infinite element ∞ such that $\infty^2 + \infty = \infty$, hence, $1 = \infty + 1 = \infty$. So, $d^{-1} + 1 = 1$ and, hence, $d = 1 + d = 1$ for every $0 \neq d \in D$; therefore, $D = \{0, 1\} \cong \mathbf{B}$.

\impliedby . If D is a division ring, then S is a semisimple ring and the statement is obvious. Let $n = 1, 2$, $D \cong \mathbf{B}$, and M be a cyclic left $M_2(\mathbf{B})$ -semimodule with an $M_2(\mathbf{B})$ -surjection $f : M_2(\mathbf{B}) \twoheadrightarrow M$. By [24, Lemma 4.7], $F(f) : \mathbf{B}^2 \cong F(M_2(\mathbf{B})) \twoheadrightarrow F(M)$ is a \mathbf{B} -surjection and, hence, $\mathbf{B}^2 / \equiv_{F(f)} \cong F(M)$. It is clear that $\{(0, 0)\}$, \mathbf{B} , $\{(0, 0), (1, 0), (1, 1)\}$ and \mathbf{B}^2 are, up to isomorphism, the only quotient semimodules of \mathbf{B}^2 which are injective by [7, Theorem 4]. Whence, $F(M)$ is an injective left \mathbf{B} -semimodule too, and therefore, by Lemma 3.8, $M \cong G(F(M))$ is an injective left $M_2(\mathbf{B})$ -semimodule as well. \square

Using this proposition, we obtain a complete description of semisimple CI-semirings:

Theorem 4.10 *A semisimple semiring S is a left (right) CI-semiring iff $S \cong S_1 \times \cdots \times S_r$, where each S_i , $i = 1, \dots, r$, is either an Artinian simple ring, or isomorphic to $M_n(\mathbf{B})$ with $n = 1, 2$.*

Proof \implies . Let $M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$ be a direct product representation $(*)$ of S . By [24, Lemma 5.2], $M_{n_i}(D_i)$, $i = 1, \dots, r$, are left CI-semirings, and, by Proposition 4.9, each D_i is either a division ring or $D_i \cong \mathbf{B}$ with $n = 1, 2$.

\impliedby . This implication follows right away from Proposition 4.9 and the obvious fact that a finite direct product of left (right) CI-semirings is also a left (right) CI-semiring. \square

A complete description of anti-bounded CI-semirings constitutes our next main goal in this paper. We need first to justify the following important and useful facts about some semirings, in particular about the ones introduced in Example 3.7 above.

Fact 4.11 *The semiring \mathbf{B}_3 , defined on the chain $0 < 1 < 2$ in Example 3.7, is a CI-semiring.*

Proof First note that, up to isomorphism, there are only two nonzero cyclic \mathbf{B}_3 -semimodules, namely $\{0, 2\}$ and \mathbf{B}_3 . Indeed, let $M = \mathbf{B}_3 m$ be a nonzero cyclic \mathbf{B}_3 -semimodule for some $0_M \neq m \in M$. Then, $2m = 0_M$ implies $0_M = 2m = (1 + 2)m = m + 2m = m + 0_M = m$ which contradicts $0_M \neq m$; if $m = 2m$ or $m \neq 2m \neq 0$, it is easy to see that then $M \simeq \{0, 2\}$ or $M \simeq \mathbf{B}_3$, respectively.

Now, considering the category of semimodules $\mathbf{B}_3 \mathcal{M}$ over the additively idempotent commutative semiring \mathbf{B}_3 , noting that the regular semimodule $\mathbf{B}_3 \mathbf{B}_3 \in |\mathbf{B}_3 \mathcal{M}|$ is a free, and therefore, a flat semimodule, and applying [21, Theorem 3.9 and Proposition 4.1], one obtains that the ‘character’ semimodule $\mathbf{2}^{\mathbf{B}_3} := \mathcal{M}(\mathbf{B}_3, \mathbf{2}) \in |\mathbf{B}_3 \mathcal{M}|$ (where $\mathbf{2}$ is the two-element semilattice) of the semimodule $\mathbf{B}_3 \mathbf{B}_3$ is an injective semimodule. We conclude the proof by noting that the semimodules $\mathbf{2}^{\mathbf{B}_3}$ and $\mathbf{B}_3 \mathbf{B}_3$ are obviously isomorphic, and that the \mathbf{B}_3 -semimodules $\{0, 2\}$ is a retract of $\mathbf{B}_3 \mathbf{B}_3$. \square

Fact 4.12 *The semiring \mathbf{B}_4 , defined on the chain $0 < 1 < 2 < 3$ in Example 3.7, is not a CI-semiring.*

Proof We shall show that the regular semimodule $\mathbf{B}_4 \mathbf{B}_4 \in |\mathbf{B}_4 \mathcal{M}|$ is not injective. Indeed, consider the semimodule $M \in |\mathbf{B}_4 \mathcal{M}|$ defined on the chain $0 < a < b < c < d$ as follows:

$$\begin{aligned} 0m &= 0 \text{ and } 1m = m \text{ for all } m \in M; \\ 2a &= a, 2b = c, 2c = c, 2d = d, 3a = 3b = 3c = 3d = d. \end{aligned}$$

Consider the subsemimodule $K = \{0, b, c, d\} \leq \mathbf{B}_4 M$ and the homomorphism $\varphi : K \rightarrow \mathbf{B}_4$ such that $0 \mapsto 0$, $b \mapsto 1$, $c \mapsto 2$ and $d \mapsto 3$. There is no extension $\tilde{\varphi} : M \rightarrow \mathbf{B}_4$ of the homomorphism φ : Indeed, if such an extension exists, we would have $\tilde{\varphi}(a) \leq \varphi(b) = 1$; then, for $\tilde{\varphi}(a) = 0$ would imply $3 = \varphi(d) = \varphi(3a) = 3\tilde{\varphi}(a) = 0$ and $\tilde{\varphi}(a) = 1$ would imply $1 = \tilde{\varphi}(a) = \tilde{\varphi}(2a) = 2\tilde{\varphi}(a) = 2$, yielding a contradiction in both cases. Therefore, the semimodule $\mathbf{B}_4 \mathbf{B}_4$ is not injective and, consequently, \mathbf{B}_4 is not a CI-semiring. \square

Fact 4.13 *The semiring $B(3, 1) = (\{0, 1, 2\}, \oplus, \odot)$ with the operations $a \oplus b \stackrel{\text{def}}{=} \min(2, a + b)$ and $a \odot b \stackrel{\text{def}}{=} \min(2, ab)$ is not a CI-semiring.*

Proof Clearly, $B(3, 1)$ is a commutative zerosumfree anti-bounded semiring (see also [10, Example 1.8]). Extend the monoid $(\{0, 1, 2\}, +, 0)$ to a commutative monoid $M = \{0, 1, 2, a_1, a_2, a_3, b_1, b_2, b_3\}$ with addition “+” defined as follows: $a_i + a_i = a_i + b_i = b_i + b_i = b_i$, $a_i + a_j = 1$ for $i \neq j$, and $x + y = 2$ for all non-zero $x, y \in M$ in any other case. One can readily verify that actually M can be naturally considered as a $B(3, 1)$ -semimodule containing ${}_{B(3,1)}B(3, 1) \leq {}_{B(3,1)}M \in |{}_{B(3,1)}\mathcal{M}|$ as a subsemimodule. However, we shall show that ${}_{B(3,1)}B(3, 1)$ is not a retract of ${}_{B(3,1)}M$.

Suppose that there exists a homomorphism $\varphi : M \rightarrow B(3, 1)$ which extends $1_{B(3,1)}$. Then, $\varphi(a_i) = 0$ or $\varphi(a_i) = 1$ for each $i \in \{1, 2, 3\}$: Indeed, otherwise there exists a_i such that $\varphi(a_i) = 2$, and $1 = \varphi(1) = \varphi(a_i + a_j) = \varphi(a_i) \oplus \varphi(a_j) = 2 \oplus \varphi(a_j) = 2$ for each $j \neq i$. If $\varphi(a_j) = 0 = \varphi(a_k)$ for at least two different indices j and k , then we have a contradiction: $1 = \varphi(1) = \varphi(a_j + a_k) = \varphi(a_j) \oplus \varphi(a_k) = 0 \oplus 0 = 0$; if $\varphi(a_j) = 1 = \varphi(a_k)$ for at least two different indices j and k , we have a contradiction: $1 = \varphi(1) = \varphi(a_j + a_k) = \varphi(a_j) \oplus \varphi(a_k) = 1 \oplus 1 = 2$. Thus, there is no $\varphi : M \rightarrow B(3, 1)$ extending $1_{B(3,1)}$; therefore, the semimodule $_{B(3,1)}B(3, 1)$ is not injective and $B(3, 1)$ is not a CI-semiring. \square

Proposition 4.14 *Every anti-bounded left (right) CI-semiring is an additively regular semiring.*

Proof Let S be an anti-bounded left (right) CI-semiring. By Proposition 4.1, $S = R \oplus T$ with R and T a semisimple ring and a zerosumfree left (right) CI-semiring, respectively. It is easy to see that R and T , being factor semirings of S (see also [24, Lemma 5.2]), are a left (right) CI-ring and zerosumfree anti-bounded CI-semiring, respectively. Therefore, it is enough to show that the semiring T is an additively regular semiring.

Suppose that a zerosumfree anti-bounded left (right) CI-semiring T with the multiplicative identity $1 \in T$ is not additively regular, *i.e.*, $1 + x + 1 \neq 1$ for all $x \in T$. Consider the congruence $\rho \in \text{Cong}(T)$ defined as follows: $a \rho b$ iff $a = b$ or there exist $x, y \in T$ such that $a = 2 + x$ and $b = 2 + y$ (notice that $2 := 1 + 1 \neq 1$ by our hypothesis). T/ρ is a left (right) CI-semiring. On the other hand, it is easy to see that $T/\rho \simeq B(3, 1)$ which, by Fact 4.13, is not a CI-semiring. Therefore, T is an additively regular semiring. \square

The next result describes all CI-semirings among additively idempotent anti-bounded semirings.

Proposition 4.15 *An additively idempotent anti-bounded semiring S is a left (right) CI-semiring iff $S \simeq \mathbf{B}$ or $S \simeq \mathbf{B}_3$.*

Proof \implies . Since S is additively idempotent, the additive monoid $(S, +, 0)$ can be partially ordered by setting $x \leq y$ iff $x + y = y$. Also, by [1, Theorem 2.2], S is a regular semiring, and hence, for each nonzero $a \in S$ there exists a nonzero $x \in S$ such that $axa = a$. Since S is anti-bounded, $x = 1 + s$, $a = 1 + s'$ for some $s, s' \in S$ and, therefore, $a = axa = a(1 + s)a = a^2 + asa \geq a^2 = a(1 + s') = a + as' \geq a$ and, hence, $a^2 = a$. Similarly, $ab \geq a$ and $ab \geq b$ and, hence, $ab \geq a + b$ for all nonzero $a, b \in S$. On the other hand, $a + b = (a + b)^2 = a^2 + ab + ba + b^2 = a + ab + ba + b \geq ab$ and, hence, $ab = a + b$. In particular, $a = a + 1$ and, hence, $1 \leq a$ for all $0 \neq a \in S$. Next, by Proposition 4.2, S has an infinite element ∞ , *i.e.*, $a + \infty = \infty$ for all $a \in S$, and, therefore, there are the following cases to consider.

If $\infty = 1$ or $S = \{0, 1, \infty\}$, then one obviously has $S = \{0, 1\} \simeq \mathbf{B}$ or $S = \{0, 1, \infty\} \simeq \mathbf{B}_3$, respectively.

Now suppose that there exists an element $a \in S$ such that $1 < a < \infty$. If there are no other elements in S , then $S \simeq \mathbf{B}_4$ and, by Fact 4.12, S is not a CI-semiring.

Finally, if there exists $b \in S \setminus \{a\}$ such that $1 < b < \infty$, we may assume, without loss of generality, that $b \not\leq a$ and consider the congruence $\rho \in \text{Cong}(S)$ defined as follows: $x\rho y$ iff $x = y$, or $1 < x \leq a$ and $1 < y \leq a$, or $x \not\leq a$ and $y \not\leq a$. Then one can easily verify that $S/\rho \simeq \mathbf{B}_4$ and, hence, by Fact 4.12, it should be simultaneously not a CI-semiring and a CI-semiring. So, this case is impossible.

\Leftarrow . This follows from Theorem 4.3 and Fact 4.11. \square

Our next observation, in fact, provides a powerful method of constructing a bunch of interesting anti-bounded semirings arising from arbitrary rings.

Example 4.16 Let $R = (R, +, \cdot, e, 1)$ be an arbitrary ring with zero e and unit 1 . Let $T := R \cup \{0\}$ and extend the operations on R to T by setting $0 + t = t = t + 0$ and $0 \cdot t = 0 = t \cdot 0$ for all $t \in T$. Clearly, $(T, +, \cdot, 0, 1)$ is a zerosumfree semiring. Now, extend the semiring structure on T to a semiring structure on $\text{Ext}(R) := T \cup \{\infty\} = R \cup \{0, \infty\}$, where $\infty \notin T$, by setting $x + \infty = \infty = \infty + \infty = \infty + x$ and $x \cdot \infty = \infty = \infty \cdot \infty = \infty \cdot x$ for all $x \in R$, and $0 \cdot \infty = 0 = \infty \cdot 0$. It is easy to see that $(\text{Ext}(R), +, \cdot, 0, 1)$ is, indeed, an anti-bounded zerosumfree semiring. In a similar fashion, one can naturally extend the structure of every left R -module M to a structure of an $\text{Ext}(R)$ -semimodule on $\text{Ext}(M)$.

Proposition 4.17 *The cyclic left (right) $\text{Ext}(R)$ -semimodules are, up to isomorphism, $\{0\}$, $\{0, \infty\}$, and $\{\text{Ext}(\overline{R}) \mid \overline{R} = R/I, \text{ where } I \text{ is a left (right) ideal of } R\}$.*

Proof Let $C \in |\text{Ext}(R)\mathcal{M}|$ be a nonzero cyclic left $\text{Ext}(R)$ -semimodule, i.e., $C = \text{Ext}(R)c$ for some $c \in C$. If there exists an element $q \in R$ such that $qc = \infty c$, then

$$q'c = (q' + e)c = qc + (q' - q)c = \infty c + (q' - q)c = (\infty + (q' - q))c = \infty c$$

for any $q' \in R$; hence $sc = \infty c$ for all $0 \neq s \in \text{Ext}(R)$ and C is isomorphic to the $\text{Ext}(R)$ -semimodule $\{0, \infty\}$.

Otherwise, for every $q \in R$, we have $qc \neq \infty c$, and so $qc \neq 0$: Indeed, $qc = 0$ implies $\infty c = \infty qc = 0$, and we get a contradiction: $0 = \infty c = (\infty + 1)c = \infty c + c = c$. Thus, $C = \{0\} \cup T \cup \{\infty c\}$ where $T = \{qc \mid q \in R\}$, is a cyclic left R -module. Whence, $C = \text{Ext}(\overline{R})$ where $\overline{R} = R/I$, for some left ideal $I \subseteq R$.

\square

Our next result gives a characterization of semirings $\text{Ext}(R)$, R is a ring, that are CI-semirings.

Theorem 4.18 *For a ring $R = (R, +, \cdot, e, 1)$, the semiring $\text{Ext}(R)$ is a left (right) CI-semiring iff R is a semisimple ring.*

Proof \implies . Let M be a cyclic left R -module, $A \leq_R B$ for $A, B \in |_R\mathcal{M}|$, and $\varphi : A \longrightarrow M$ be an R -homomorphism. Then, $\text{Ext}(M)$ is a cyclic left $\text{Ext}(R)$ -semimodule and φ induces an $\text{Ext}(R)$ -homomorphism $\psi : \text{Ext}(A) \longrightarrow \text{Ext}(B)$. Since $\text{Ext}(R)$ is a left CI-semiring, the latter can be extended to $\text{Ext}(R)$ -homomorphism $\tilde{\psi} : \text{Ext}(B) \longrightarrow \text{Ext}(M)$. It is easy to see that the restriction $\tilde{\varphi} := \tilde{\psi}|_B : B \longrightarrow M$ is an R -homomorphism extending φ . Consequently, R is a CI-ring and, hence, by [38, p. 649] (see also [34, Theorem 1.2.9]), R is a semisimple ring.

\Leftarrow . Let $R = (R, +, \cdot, e, 1)$ be a semisimple ring, $\text{Ext}(R) = R \cup \{0, \infty\}$, $M = \{\bar{0}\} \cup \bar{R} \cup \{\infty\}$ a cyclic $\text{Ext}(R)$ -semimodule with $\bar{R} = R/I$, where I is a left (right) ideal of R , and ${}_{\text{Ext}(R)}M \leq {}_{\text{Ext}(R)}T \in |_{\text{Ext}(R)}\mathcal{M}|$. Since $\text{Ext}(R)$ is an additively regular semiring, an additive reduct of every $\text{Ext}(R)$ -semimodule is a commutative inverse monoid as well. Let $M = [Y; G_\alpha, \varphi_{\alpha, \beta}]$ and $T = [Z; H_\alpha, \psi_{\alpha, \beta}]$ be *Clifford representations* of the monoids $(M, +, 0)$ and $(T, +, 0)$ (see [6, Theorem 4.11] or [39, Theorem II.2.6], also cf. [21, p. 125 and Proposition 2.4]), respectively. As usual, it is convenient to identify elements of the semilattices Y and Z in the Clifford representations with the zeros of the corresponding abelian groups. Then, it is clear that $Y = \{\bar{0}, \bar{e}, \infty\} \cong \mathbf{B}_3$, $Y \subseteq Z$ and $\psi_{\alpha, \beta}(a) = a + \beta$ for any $a \in H_\alpha$. Also, it is easy to see that $a + b \in H_{\alpha + \beta}$ and $sa \in H_{s\alpha}$ for every $a \in H_\alpha$, $b \in H_\beta$, and $s \in \text{Ext}(R)$; in particular, since $r\alpha = \alpha$ for all $r \in R$ and $\alpha \in Z$, one has $\psi_{\alpha, \beta}(ra) = ra + \beta = ra + r\beta = r(a + \beta) = r\psi_{\alpha, \beta}(a)$ for all $r \in R$ and $a \in H_\alpha$, i.e., $\psi_{\alpha, \beta}$ are R -homomorphisms. On the semilattices Y and Z , of course, there exists a natural partial ordering defined as follows: $\alpha \leq \beta$ iff $\alpha + \beta = \beta$.

Since R is a semisimple ring, by [38] (see also [34, Theorem 1.2.9] or [33, Corollary 6.47]), there exists an R -homomorphism $\Theta : H_{\bar{e}} \longrightarrow G_{\bar{e}}$ extending the identity R -isomorphism $1_{G_{\bar{e}}} : G_{\bar{e}} \longrightarrow G_{\bar{e}}$. We shall show that the map $\Theta^* : T \longrightarrow M$ for $a \in H_\alpha$, defined by

$$\Theta^*(a) := \begin{cases} \bar{0}, & \text{if } \infty\alpha \leq \bar{e}, \\ \Theta\psi_{\alpha, \beta}, & \text{if } \alpha \leq \bar{e} \text{ \& } \infty\alpha \not\leq \bar{e}, \\ \infty & \text{in all other cases} \end{cases},$$

is an $\text{Ext}(R)$ -homomorphism extending the identity $\text{Ext}(R)$ -homomorphism $1_M : M \longrightarrow M$, and therefore, M is a retract of $T \in |_{\text{Ext}(R)}\mathcal{M}|$.

First, considering all possible cases, we need to verify that $\Theta^*(a+b) = \Theta^*(a) + \Theta^*(b)$ for all $a \in H_\alpha$ and $b \in H_\beta$.

If $\Theta^*(a) = \infty$, then $\alpha \not\leq \bar{e}$ and, hence, $a + \beta \not\leq \bar{e}$; whence $\Theta^*(a+b) = \infty = \infty + \Theta^*(b) = \Theta^*(a) + \Theta^*(b)$. The case when $\Theta^*(b) = \infty$ can be similarly justified.

If $\Theta^*(a), \Theta^*(b) \in G_{\bar{e}}$, then $\alpha \leq \bar{e}, \beta \leq \bar{e}$ and, hence, $\alpha + \beta \leq \bar{e}$ and $\infty(\alpha + \beta) \not\leq \bar{e}$; whence, $\Theta^*(a+b) = \Theta(a+b+\bar{e}) = \Theta(a+\bar{e}+b+\bar{e}) = \Theta(a+\bar{e}) + \Theta(b+\bar{e}) = \Theta^*(a) + \Theta^*(b)$.

If $\Theta^*(a) = \bar{0} = \Theta^*(b)$, then $\infty\alpha \leq \bar{e}, \infty\beta \leq \bar{e}$, and, hence, $\infty(\alpha + \beta) \leq \bar{e}$; whence, $\Theta^*(a+b) = \bar{0} = \bar{0} + \bar{0} = \Theta^*(a) + \Theta^*(b)$.

If $\Theta^*(a) = \bar{0}$ and $\Theta^*(b) \in G_{\bar{e}}$, then $\infty\alpha \leq \bar{e}, \beta \leq \bar{e}, \infty\beta \not\leq \bar{e}$, and, therefore, $\alpha + \beta \leq \infty\alpha + \beta = \beta \leq \bar{e}$, and $\infty(\alpha + \beta) \not\leq \bar{e}$. Clearly, $\infty a = \infty\alpha$ and so $a + b + \bar{e} = a + b + \bar{e} + \infty\alpha = a + b + \bar{e} + \infty a = b + \bar{e} + (1 + \infty)a = b + \bar{e} + \infty a = b + \bar{e}$. Therefore, $\Theta^*(a+b) = \Theta(a+b+\bar{e}) = \Theta(b+\bar{e}) = \bar{0} + \Theta(b+\bar{e}) = \Theta^*(a) + \Theta^*(b)$. Of course, the case $\Theta^*(b) = \bar{0}$ and $\Theta^*(a) \in G_{\bar{e}}$ is justified in a similar way.

Now, considering all possible cases, we need to verify that $\Theta^*(sa) = s\Theta^*(a)$ for all $a \in H_\alpha$ and $s \in \text{Ext}(R)$, where, certainly, we may assume $s \neq 0$.

So, if $\Theta^*(a) = \bar{0}$, then $\infty(sa) = \infty\alpha \leq \bar{e}$ and so $\Theta^*(sa) = \bar{0} = s\bar{0} = s\Theta^*(a)$.

If $\Theta^*(a) \in G_{\bar{e}}$ and $s \in R$, then $\Theta^*(sa) = s\Theta^*(a)$ is true since the composite $\Theta\psi_{\alpha, \bar{e}}$ is obviously an R -homomorphism as well.

If $\Theta^*(a) \in G_{\bar{e}}$ and $s = \infty$, then $\infty a \in H_{\infty\alpha}$ and $\infty\alpha \not\leq \bar{e}$ and so $\Theta^*(\infty a) = \bar{\infty} = \infty\Theta^*(a)$.

If $\Theta^*(a) = \bar{\infty}$, then $\alpha \not\leq \bar{e}$ and, since $\alpha \leq s\alpha$ for $s \neq 0$, one gets $s\alpha \not\leq \bar{e}$, and therefore, $\Theta^*(sa) = \bar{\infty} = s\bar{\infty} = s\Theta^*(a)$.

Thus, we have shown that $\Theta^* : T \rightarrow M$ is an $\text{Ext}(R)$ -homomorphism extending the identity $\text{Ext}(R)$ -homomorphism $1_M : M \rightarrow M$, and therefore, M is a retract of $T \in |\text{Ext}(R)\mathcal{M}|$. Now, taking into consideration that $\text{Ext}(R)$ is an additively regular semiring and applying [21, Theorem 4.2 or Corollary 4.3], we can choose the semimodule $T \in |\text{Ext}(R)\mathcal{M}|$ to be injective and, hence, conclude that M is an injective $\text{Ext}(R)$ -semimodule, too. To finish the proof, one needs only to use Proposition 4.17 and note that the $\text{Ext}(R)$ -semimodule $\{0, \infty\}$ is obviously a retract of the regular semimodule $_{\text{Ext}(R)}\text{Ext}(R)$. \square

In the next observation, and as a consequence of the previous theorem, we obtain a complete description of zerosumfree additively regular anti-bounded CI-semirings.

Proposition 4.19 *A zerosumfree additively regular anti-bounded semiring S is a left (right) CI-semiring iff $S \simeq \mathbf{B}$, or $S \simeq \mathbf{B}_3$, or $S \simeq \text{Ext}(R)$ for some nonzero semisimple ring R .*

Proof \implies . By Proposition 4.1, S contains an infinite element ∞ . Consider the congruence $\diamond \in \text{Cong}(S)$ defined as follows: $s \diamond s'$ iff $ns = s' + t$ and $ms' = s + t'$ for some $n, m \in \mathbb{N}$ and $t, t' \in S$. By [19, Lemma 2.2], the quotient semiring $\bar{S} = S/\diamond$ is an additively idempotent, left (right) CI-semiring. Thus, applying Proposition 4.15, $\bar{S} \simeq \mathbf{B}$ or $\bar{S} \simeq \mathbf{B}_3$, and we will consider each of these cases.

If $\bar{S} \cong \mathbf{B}$, then we have $1 \diamond \infty$ and, hence, $n1 = \infty + s = \infty$ for some $n \in \mathbb{N}$ and $s \in S$. Since S is an additively regular semiring, there exists $x \in S$ such that

$1 = 1 + x + 1$ and, therefore, $1 = n(1 + x) + 1 = n1 + nx + 1 = \infty + nx + 1 = \infty$. Whence, S is even an additively idempotent semiring and, clearly, $S \cong \overline{S} \simeq \mathbf{B}$.

Now, let $\overline{S} \cong \mathbf{B}_3$, and $x \in S$ be an additive inverse of $1 \in S$, i.e., $1 + x + 1 = 1$. Then, for the element $e := 1 + x \in I^+(S)$ and all $s \in S$ and $a \in I^+(S)$, we have

$$e^2 = (1 + x)^2 = 1 + x + x + x^2 = 1 + x(1 + 1 + x) = 1 + x \cdot 1 = 1 + x = e,$$

$$\begin{aligned} se &= s + sx = (1 + 1 + x)s + sx = s + s + xs + sx = xs + s(1 + 1 + x) = \\ xs + s &= es \text{ and} \end{aligned}$$

$$a = a \cdot 1 = a(1 + 1 + x) = a + a + ax = a + ax = a(1 + x) = ae.$$

It is clear that $e \neq 0$, $e \neq \infty$ and the restriction $\pi|_{I^+(S)}$ of the natural surjection $\pi : S \longrightarrow \overline{S}$ is an injection. Therefore, $I^+(S) = \{0, e, \infty\}$.

As was shown above for $a = 1$, it can be shown that $a \diamond \infty$ implies $a = \infty$, and therefore, the equivalence classes 0^\diamond and ∞^\diamond are $\{0\}$ and $\{\infty\}$, respectively. Thus, $S = \{0\} \cup R \cup \{\infty\}$, where R is the equivalence class e^\diamond . If $|R| = 1$, then it is easy to see that $S \cong \mathbf{B}_3$. Therefore, we have only to consider the case when $|R| > 1$. Obviously, R is closed under addition and multiplication. Hence, for any $a \in R$ we have $ae \in R \cap I^+(S)$ and so $ae = e$ and, consequently, $a + e = a + ae = a(1 + e) = a \cdot 1 = a$ and $a + ax = a(1 + x) = ae = e$. Whence, $(R, +, \cdot, e, 1)$ is a ring and $S = \text{Ext}(R)$ and, by Theorem 4.18, R is a semisimple ring.

\Leftarrow . This follows from Proposition 4.15 and Theorem 4.18. \square

Applying Propositions 4.1, 4.15 and 4.19, we obtain a complete characterization of anti-bounded CI-semirings generalizing Osofsky's celebrated characterization of semisimple rings ([38], see also [34, Theorem 1.2.9] and [33, Corollary 6.47]):

Theorem 4.20 *An anti-bounded semiring S is a left (right) CI-semiring iff S is one of the following semirings:*

- (1) S is a semisimple ring;
- (2) $S \simeq \mathbf{B}$, or $S \simeq \mathbf{B}_3$, or $S \simeq \text{Ext}(R)$ for some nonzero semisimple ring R ;
- (3) $S = R \oplus T$, where R is a semisimple ring and T is isomorphic to \mathbf{B} , or \mathbf{B}_3 , or $\text{Ext}(R')$ for some nonzero semisimple ring R' .

Finally, as an application of Theorem 4.20, we solve [25, Problem 3] and [27, Problem 5] in the class of anti-bounded CI-semirings.

Corollary 4.21 *For an anti-bounded semiring S , the following conditions are equivalent:*

- (i) S is a congruence-simple left (right) CI-semiring;
- (ii) S is an ideal-simple left (right) CI-semiring;
- (iii) $S \cong M_n(D)$ for some division ring D and $n \geq 1$, or $S \cong \mathbf{B}$.

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